Let us consider an exponential function E, defined by

$$E(x) = b^x,$$
where $b > 0$ and $b \neq 1$.

The domain of E is the set of all real numbers, and the range of E is the set of *positive* real numbers. Thus, for each real number r, there exists a unique positive real number s such that $s = b^r$.

If we draw the graph of E, as shown in Fig. 7–1, then another property of the function can be seen: For each positive number s, the line $y = s$ intersects the graph of E in one, and only one, point, (r, s). This means that the range of E is the set of all positive real numbers, and that for each positive real number s, there exists a *unique* real number r such that $b^r = s$.

The remarks above lead us to define the logarithm function to the base b, denoted by $\log_b$, in the following manner.

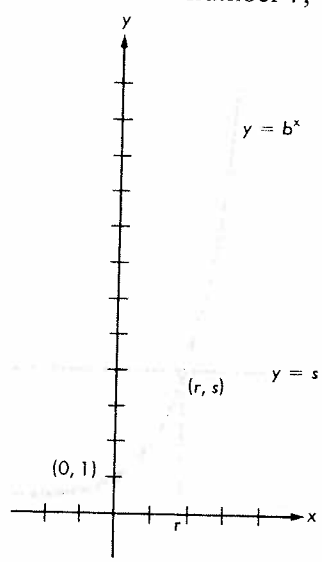


FIGURE 7–1

Definition of the Logarithm Function

For every positive number s,

$$\log_b s = r,$$

where r is the number such that $b^r = s$ (assuming $b > 0$ and $b \neq 1$).

According to this definition, the function $\log_b$ has the set of all positive real numbers as its domain. Also,

$\log_b s$ is the power to which b must be raised to give s.

To illustrate the definition, let us find each of the following logarithms.

E, ..fined by

$\neq 1$.

nbers, and the range of E is
or each real number r, there

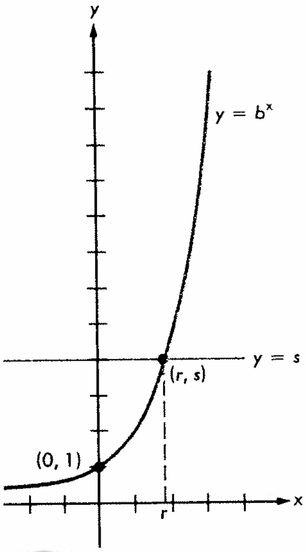


FIGURE 7-1

ssuming $b > 0$ and $b \neq 1$).

n $\log_b$ has the set of all posi-

st be raised to give s.

find each of the following

$\log_2 16$. If we let $r = \log_2 16$, then $2^r = 16$. Since $16 = 2^4$, we conclude that $r = 4$. Thus,

$$\log_2 16 = 4.$$

$\log_2 \sqrt[3]{2}$. If we let $r = \log_2 \sqrt[3]{2}$, then $2^r = \sqrt[3]{2}$. Since $\sqrt[3]{2} = 2^{\frac{1}{3}}$, we conclude that $2^r = 2^{\frac{1}{3}}$ and $r = \frac{1}{3}$. Thus,

$$\log_2 \sqrt[3]{2} = \tfrac{1}{3}.$$

$\log_2 \frac{1}{4}$ If we let $r = \log_2 \frac{1}{4}$, then $2^r = \frac{1}{4}$. Since $\frac{1}{4} = 2^{-2}$, we conclude that $r = -2$. Thus,

$$\log_2 \tfrac{1}{4} = -2.$$

$\log_5 5$. If we let $r = \log_5 5$, then $5^r = 5$. Since $5^1 = 5$, we conclude that $r = 1$ and

$$\log_5 5 = 1.$$

$\log_8 16$. If we let $r = \log_8 16$, then $8^r = 16$. To find the power of 8 that is equal to 16, we replace 8 by 2^3, obtaining the equation $(2^3)^r = 16$ or, equivalently, $2^{3r} = 16$. Since $16 = 2^4$, we conclude that $2^{3r} = 2^4$ and $3r = 4$. Thus, $r = \frac{4}{3}$ and

$$\log_8 16 = \tfrac{4}{3}.$$

$\log_{\frac{1}{10}} 1000$. If we let $r = \log_{\frac{1}{10}} 1000$, then $(\frac{1}{10})^r = 1000$. Since $\frac{1}{10} = 10^{-1}$, we conclude that $(10^{-1})^r = 1000$, or

$$10^{-r} = 1000.$$

Now $1000 = 10^3$, and, therefore, $-r = 3$ or $r = -3$. Thus,

$$\log_{\frac{1}{10}} 1000 = -3.$$

Each problem above is solved by using the fact that the two equations

$$y = \log_b x \quad \text{and} \quad b^y = x$$

are equivalent, that is, that they have the same solution set.

Problem. Solve each of the following equations.

(a) $4 = \log_5 x$ (b) $y = \log_4 32$ (c) $\frac{2}{3} = \log_b 9$

Solution.

(a) The equation $4 = \log_5 x$ is equivalent to the equation $5^4 = x$. Hence, $x = 625$ is the solution.

(b) The equation $y = \log_4 32$ is equivalent to the equation $4^y = 32$. Since $4 = 2^2$ and $32 = 2^5$, another equivalent equation is $(2^2)^y = 2^5$, or $2^{2y} = 2^5$. Hence, $2y = 5$ and $y = \frac{5}{2}$.

(c) The equation $\frac{2}{3} = \log_b 9$ is equivalent to the equation $b^{\frac{2}{3}} = 9$. In turn, this latter equation is equivalent to the equation $(b^{\frac{2}{3}})^{\frac{3}{2}} = 9^{\frac{3}{2}}$, or $b = 9^{\frac{3}{2}}$. Hence, $b = 27$ is the solution. Although it is true that $(-27)^{\frac{2}{3}} = (\sqrt[3]{-27})^2$, or 9, so that $b = -27$ is a solution of the equation $b^{\frac{2}{3}} = 9$, no negative number can be a base for logarithms. Thus, $b = 27$ is the only solution of the given equation.

The following properties of logarithms follow from the definition of a logarithm.

$$\log_b 1 = 0 \quad \text{since} \quad b^0 = 1$$
$$\log_b b = 1 \quad \text{since} \quad b^1 = b$$
$$\log_b b^r = r \quad \text{since} \quad b^r = b^r$$

$$\log_{\frac{1}{b}} x = -\log_b x \quad \text{since} \quad \left(\frac{1}{b}\right)^r = b^{-r}$$

In view of the last property, we might as well restrict the base b to be greater than 1, for if $0 < b < 1$, then $1/b > 1$, and each logarithm to the base b is easily expressed in terms of a logarithm to the base $1/b$. For example,

$$\log_{\frac{1}{2}} 4 = -\log_2 4, \quad \text{or} \quad -2,$$
$$\log_{\frac{1}{10}} .001 = -\log_{10} .001, \quad \text{or} \quad -(-3), \quad \text{or} \quad 3.$$

Exercises

Write the logarithmic equation that corresponds to each of the following exponential equations.

1. (a) $2^5 = 32$ (b) $3^{-2} = \frac{1}{9}$

2. (a) $7^0 = 1$ (b) $25^{\frac{3}{2}} = 125$

3. (a) $16^{-\frac{3}{2}} = \frac{1}{64}$ (b) $10^0 = 1$

4. (a) $16^{-\frac{3}{4}} = .125$ (b) $(\frac{1}{3})^{-2} = 9$

Write the exponential equation that corresponds to each of the following logarithmic equations.

5. (a) $\log_{10} 100 = 2$ (b) $\log_8 4 = \frac{2}{3}$

6. (a) $\log_{\frac{1}{3}} 81 = -4$ (b) $\log_{25} \frac{1}{125} = -\frac{3}{2}$

7. (a) $\log_8 1 = 0$ (b) $\log_a y = x$

8. (a) $\log_n 256 = 2$ (b) $\log_{16} 8 = \frac{3}{4}$

9. (a) $\log_n 9 = .4$ (b) $\log_{\frac{1}{8}} 4 = -\frac{2}{3}$

Find the value of each of the following logarithms.

10. (a) $\log_{10} 10^2$ (b) $\log_2 2^{10}$

11. (a) $\log_b b^4$ (b) $\log_b b$

12. (a) $\log_7 \sqrt[3]{7}$ (b) $\log_2 8\sqrt{32}$

13. (a) $\log_7 \frac{1}{49}$ (b) $\log_3 \sqrt[5]{81}$

14. (a) $\log_{10} \sqrt[3]{100}$ (b) $\log_{27} 81$

15. (a) $\log_{\frac{1}{10}} 100$ (b) $\log_{100} .001$

Solve each of the following equations for n.

16. $\log_{10} .0001 = n$ **17.** $\log_{36} n = -\frac{3}{2}$ **18.** $\log_n 125 = -\frac{3}{4}$

19. $\log_7 7^{-3} = n$ **20.** $\log_{49} n = \frac{3}{2}$ **21.** $\log_n 1000 = 1.5$

22. $\log_{16} 8 = n$ **23.** $\log_{64} n = \frac{7}{6}$ **24.** $\log_n (\frac{1}{81}) = -2$

In Exercises 25–30, solve each of the equations for x.

25. $\log_{10} 1 = x$ **26.** $\log_x 1 = 0$ **27.** $\log_4 x = -3$

28. $\log_x 3 = \frac{1}{4}$ **29.** $x = \log_{36} 216$ **30.** $x = \log_2 2$

7-3 LOGARITHMIC GRAPHS

The graph of the function $\log_b$ is simply the graph of the equation

$$y = \log_b x.$$

The particular graph of $\log_2$ is sketched in Fig. 7-2 from the following table of values.

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
y	-3	-2	-1	0	1	2	3	4

This graph is very similar to the exponential graph with base 2 sketched in Fig. 7-1. In fact, the two graphs can be made to coincide by turning over the piece of paper on which one is drawn and placing the positive x-axis of the logarithmic graph along the positive y-axis of the exponential graph. This is so because the two equations $y = \log_2 x$ and $2^y = x$ are equivalent.

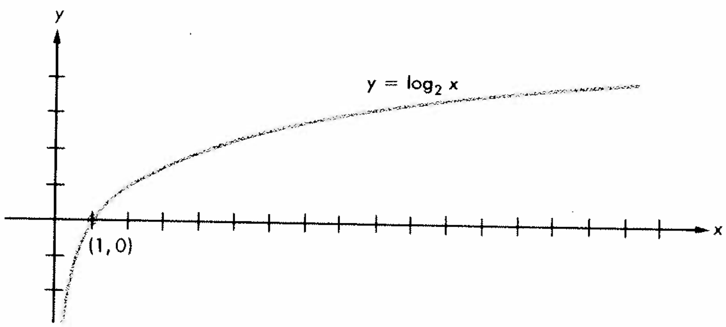


FIGURE 7-2

The logarithmic graph with base $b > 1$ has the following important properties, which are illustrated in Fig. 7-2.

1. *The entire graph is to the right of the y-axis.*
2. *The graph has x-intercept 1.*
3. *The graph is rising as we traverse it from left to right. In other words,*

$$\log_b x_1 > \log_b x_2 \quad \textit{if, and only if,} \quad x_1 > x_2.$$

The first property is true because $\log_b x$ is defined only if $x > 0$, and the second property is true because $\log_b x = 0$, if and only if, $x = 1$. The proof of the third property is too difficult to be included in this text.

Exercises

1. (a) Graph each of the following equations on the same set of axes.

$$y = \log_5 x, \quad y = 5^x$$

Is there a line of folding so that the two graphs would coincide? By what transformation could one equation be obtained from the other?

(b) Graph each of the following equations on the same set of axes.

$$y = \log_2 x, \quad y = \log_3 x, \quad y = \log_5 x, \quad y = \log_{10} x$$

What point does the following family of logarithmic curves have in common?

$$\{y = \log_b x \mid b \text{ a real number, } b > 1\}$$

Using Property 3 of the logarithmic curve with base $b > 1$, give two consecutive integers between which each of the following numbers lies.

2. (a) $\log_2 6.45$
(b) $\log_{10} 36.125$

3. (a) $\log_{10} .375$
(b) $\log_2 .3$

4. (a) If $0 < N < 1$, what can be said about $\log_{10} N$?
(b) If $N > 1$, what can be said about $\log_{10} N$?

---

5. If M is a number greater than the positive number N, how do the numbers $\log_{10} M$ and $\log_{10} N$ compare?

Graph each of the following equations.

6. $y = \log_3 x$ **7.** $y = \log_3 2x$ **8.** $y = 2 + \log_3 x$

9. $y = \log_3 (x + 2)$ **10.** $y = 2 \log_3 x$ **11.** $y = \log_3 x^2$

12. $y = \frac{1}{2} \log_3 x$ **13** $y = \log_3 \sqrt{x}$

14. Refer to Exercises 6–13 to answer the following questions.
(a) Which of the graphs are identical?
(b) Which of the graphs have the same x-intercept?
(c) For what set of values of x is each equation defined?

15. (a) Graph $y = \log_2 (-x)$.
(b) For what set of values of x is $\log_2 (-x)$ defined?

---

16. Graph $y = \log_3 (-x)$ and $y = -\log_3 x$ on the same set of axes.

17. For what set of values of x is $\log_2 (x + 1)$ defined? Graph the equation $y = \log_2 (x + 1)$. Compare the graph with that of $y = \log_2 x$.

18. (a) Sketch the graph of the equation $y = \log_5 1/x$.
(b) Graph the equation $y = -\log_5 x$.

Preparation for Section 7–4

1. What is the exponential equation equivalent to $\log_b x = r$?
2. What is the exponential equation equivalent to $\log_b y = s$?
3. Simplify $b^r \cdot b^s$.
4. Try to combine your results in the three preceding exercises to write an equation relating $\log_b xy$ to $\log_b x$ and $\log_b y$.

7-4 THE LAWS OF LOGARITHMS

The laws of exponents, which we verified first for use with positive integers and later for use with rational numbers, will now be assumed to be valid for real numbers. Since $\log_b x = y$ if, and only if, $x = b^y$, we might expect that the laws of exponents could be translated into the language of logarithms. This is the case, as we shall show below. In our development, the base b is always assumed to be a real number greater than 1.

According to the first law of exponents, the equation

$$b^{r+s} = b^r \cdot b^s$$

is true for every pair r, s of real numbers. Let us try to convert this equation into one relating logarithms by letting

$$z = b^{r+s}, \quad x = b^r, \quad y = b^s.$$

By the first law of exponents, $z = xy$. Each of the above equations can be translated into the language of logarithms, yielding the three equivalent equations

$$r + s = \log_b z, \quad r = \log_b x, \quad s = \log_b y.$$

Since $r + s$ is the sum of r and s, it follows that

$$\log_b z = \log_b x + \log_b y.$$

If we recall that $z = xy$, then we have proved the following law.

FIRST LAW OF LOGARITHMS

$$\log_b xy = \log_b x + \log_b y \qquad (LL\text{-}1)$$

This equation is true for every pair x, y of positive real numbers. The first law of logarithms can be stated in words in the following way.

The logarithm of the product of two positive numbers is the sum of the logarithms of the two numbers.

If we start with the second law of exponents and reason as we did above, then we can derive the following law.

SECOND LAW OF LOGARITHMS

$$\log_b \frac{x}{y} = \log_b x - \log_b y \qquad (LL\text{-}2)$$

This equation is also true for every pair x, y of positive real numbers. The second law of logarithms can be stated in words in the following way.

> *The logarithm of the quotient of two positive numbers is the logarithm of the dividend minus the logarithm of the divisor.*

According to the third law of exponents, the equation

$$b^{ar} = (b^a)^r$$

is true for every pair a, r of real numbers. To convert this equation to one relating logarithms, let

$$y = b^{ar} \quad \text{and} \quad x = b^a.$$

By the third law of exponents, $y = x^r$. The two equations above are equivalent to the two logarithmic equations

$$ar = \log_b y, \quad a = \log_b x.$$

Since ar equals r times a, we have

$$\log_b y = r \cdot \log_b x.$$

Replacing y by x^r, we obtain the following law.

THIRD LAW OF LOGARITHMS

$$\log_b x^r = r \cdot \log_b x \qquad (LL\text{-}3)$$

This equation is true for every real number r and every positive real number x. The third law of logarithms can be stated in words in the following way.

> *The logarithm of the rth power of a positive number is r times the logarithm of the number.*

Problem 1. Find each of the following.

(a) $\log_2 16\sqrt{8}$ (b) $\log_3 \dfrac{\sqrt[4]{27}}{9}$

Solution.

(a)
$$\begin{aligned}
\log_2 16\sqrt{8} &= \log_2 16 + \log_2 \sqrt{8} && \text{(LL-1)}\\
&= \log_2 16 + \log_2 8^{\frac{1}{2}}\\
&= \log_2 16 + \tfrac{1}{2}\log_2 8 && \text{(LL-3)}\\
&= \log_2 2^4 + \tfrac{1}{2}\log_2 2^3\\
&= 4 + (\tfrac{1}{2}\cdot 3), \quad \text{or } \tfrac{11}{2}
\end{aligned}$$

In other words, $\log_2 16\sqrt{8} = \frac{11}{2}$.

(b)
$$\begin{aligned}
\log_3 \frac{\sqrt[4]{27}}{9} &= \log_3 \sqrt[4]{27} - \log_3 9 && \text{(LL-2)}\\
&= \log_3 27^{\frac{1}{4}} - \log_3 9\\
&= \tfrac{1}{4}\log_3 27 - \log_3 9 && \text{(LL-3)}\\
&= \tfrac{1}{4}\log_3 3^3 - \log_3 3^2\\
&= (\tfrac{1}{4}\cdot 3) - 2, \quad \text{or } -\tfrac{5}{4}
\end{aligned}$$

In other words,

$$\log_3 \frac{\sqrt[4]{27}}{9} = -\frac{5}{4}.$$

Problem 2. Express the following number as the logarithm of a single number.

$$3\log_5 4 - 2\log_5 6 + \tfrac{3}{2}\log_5 18$$

Solution. We proceed as follows:

$$\begin{aligned}
3\log_5 4 - 2\log_5 6 + \tfrac{3}{2}\log_5 18\\
&= \log_5 4^3 - \log_5 6^2 + \log_5 18^{\frac{3}{2}} && \text{(LL-3)}\\
&= \log_5 64 - \log_5 36 + \log_5 (\sqrt{18})^3\\
&= \log_5 \tfrac{64}{36} + \log_5 (3\sqrt{2})^3 && \text{(LL-2)}\\
&= \log_5 \tfrac{16}{9} + \log_5 54\sqrt{2}\\
&= \log_5 (\tfrac{16}{9}\cdot 54\sqrt{2}) && \text{(LL-1)}\\
&= \log_5 (96\sqrt{2}).
\end{aligned}$$

Problem 3. Solve the equation $\log_4 3 + \log_4 (x + 2) = 2$.

Solution. Using the first law of logarithms, we have

$$\log_4 3 + \log_4 (x + 2) = \log_4 3(x + 2).$$

Hence, the given equation is equivalent to the equation

$$\log_4 3(x + 2) = 2.$$

This logarithmic equation is equivalent to the exponential equation

$$3(x + 2) = 4^2.$$

Hence,

$$3x + 6 = 16 \quad \text{and} \quad x = \tfrac{10}{3}.$$

Exercises

Find the value of each of the following by using the laws of logarithms.

1. (a) $\log_2 \sqrt[3]{32}$ (b) $\log_3 \left(\frac{\sqrt[5]{9}}{3}\right)$

2. (a) $\log_5 \left(\frac{\sqrt[3]{25}}{\sqrt{5}}\right)$ (b) $\log_{10} (10\sqrt[3]{100})$

3. (a) $\log_5 (25 \cdot 125)$ (b) $\log_6 \sqrt{216}$

4. (a) $\log_3 \sqrt[4]{9^5}$ (b) $\log_7 (49 \div 7^5)$

5. (a) $\log_3 81\sqrt{27}$ (b) $\log_2 \frac{\sqrt[4]{8}}{4}$

Express each of the following numbers as the logarithm of a single number.

6. (a) $\log_5 6 - \log_5 2$ (b) $\log_5 80 + \log_5 \frac{1}{4}$

7. (a) $4 \log_3 10 - 2 \log_3 5$ (b) $5 \log_7 9 - 4 \log_7 15 + \frac{3}{2} \log_7 12$

If $\log_{10} 2 = p$ and $\log_{10} 3 = q$, find an expression in terms of p and q for each logarithm in Exercises 8–13.

8. $\log_{10} 4$ **9.** $\log_{10} (\frac{2}{3})$

10. $\log_{10} \sqrt{2}$ **11.** $\log_{10} .5$

12. $\log_{10} 30$ **13.** $\log_{10} \frac{1}{3}$

If $\log_{10} 5 = r$ and $\log_{10} 7 = s$, find an expression for each of the following in terms of r and s.

14. $\log_{10} 35$ **15.** $\log_{10} \frac{7}{5}$

16. $\log_{10} 49$ **17.** $\log_{10} \frac{343}{25}$

18. $\log_{10}(\sqrt{7}\cdot\sqrt[3]{5})$

19. $\log_{10}\frac{7}{10}\cdot\log_{10}125$

20. $\log_{10}50-\log_{10}700$

21. $\log_{10}\sqrt{\frac{7}{5}}$

Solve each of the following equations.

22. $\log_4 72-\log_4 9=N$

23. $\log_5 6=\log_5 x-\log_5 7$

24. $\log_7 98+\log_7 3.5=N$

25. $\log_3 63-\log_3 7x=\log_3 2$

26. $\log_8\sqrt{.125}=x$

27. $\log_5 4+\log_5(2x-3)=20$

28. Prove the second law of logarithms, (LL-2).

In Exercises 29–31, use the laws of logarithms to prove that each of the equations is true.

29. $\log_b 16-\log_b 8+\log_b 5=\log_b 10$

30. $2\log_x a-2\log_x b+3\log_x\sqrt{b}-\frac{1}{3}\log_x a=\frac{1}{6}\log_x\frac{a^{10}}{b^3}$

31. $\log_3\frac{x^2 3^x}{3^{x^2}}=x-x^2+2\log_3 x$

32. Solve the following equation for x.

$$\log_2 2+\log_2(x+2)-\log_2(3x-5)=3$$

33. Prove that

$$\log_{10}\frac{3x-\sqrt{9x^2-1}}{3x+\sqrt{9x^2-1}}=2\log_{10}(3x-\sqrt{9x^2-1}).$$

34. Solve each of the following equations.
    (a) $y=2\log_2 8$
    (b) $y=3\log_3 81$
    (c) $y=10\log_2 4$
    (d) $-y=b\log_b b^5$

Review for Sections 7–1 through 7–4

Write the logarithmic equation that corresponds to each of the following exponential equations.

1. $4^3=64$

2. $9^0=1$

3. $8^{\frac{2}{3}}=4$

4. $49^{\frac{3}{2}}=343$

5. $10^{-x}=y$

6. $a^{-\frac{3}{2}}=c$

Write the exponential equation that corresponds to each of the following logarithmic equations.

7. $\log_6 6=1$

8. $\log_3 243=5$

9. $\log_4\frac{1}{2}=-\frac{1}{2}$

10. $\log_5\frac{1}{25}=-2$

11. $\log_8 y=2x$

12. $\log_7 3y=x$

Solve each of the following equations for x.

13. $\log_3 27 = x$

14. $\log_4 64 = x$

15. $\log_{16} x = \frac{1}{2}$

16. $\log_{25} x = \frac{3}{2}$

17. $\log_x \frac{1}{49} = -2$

18. $\log_x \frac{1}{81} = -4$

Graph each of the following and name the x-intercept.

19. $y = \log_5 x$

20. $y = \log_5 (x + 1)$

21. $y = \frac{1}{2} \log_5 x$

22. $y = 2 \log_5 x$

Solve each of the following equations.

23. $y = \frac{1}{2} \log_3 81$

24. $y = -\frac{1}{2} \log_2 \frac{1}{64}$

25. $y = \log_2 2\sqrt{2}$

26. $y = \log_3 9\sqrt{27}$

Answers to Review for Sections 7–1 through 7–4

1. $\log_4 64 = 3$

2. $\log_9 1 = 0$

3. $\log_8 4 = \frac{2}{3}$

4. $\log_{49} 343 = \frac{3}{2}$

5. $\log_{10} y = -x$

6. $\log_a c = -\frac{3}{2}$

7. $6^1 = 6$

8. $3^5 = 243$

9. $4^{-\frac{1}{2}} = \frac{1}{2}$

10. $5^{-2} = \frac{1}{25}$

11. $8^{2x} = y$

12. $7^x = 3y$

13. $x = 3$

14. $x = 3$

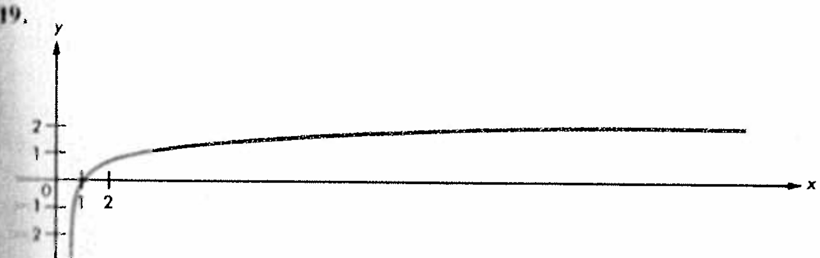
15. $x = 4$

16. $x = 125$

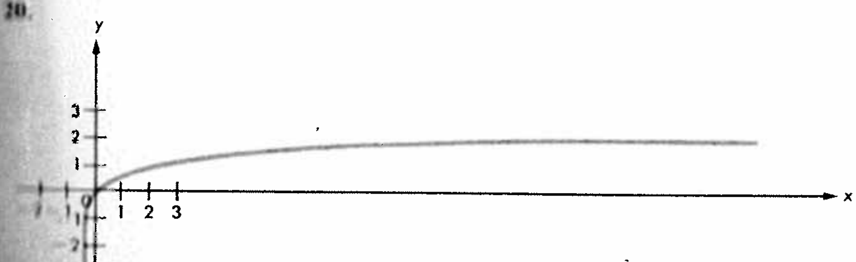
17. $x = 7$

18. $x = 3$

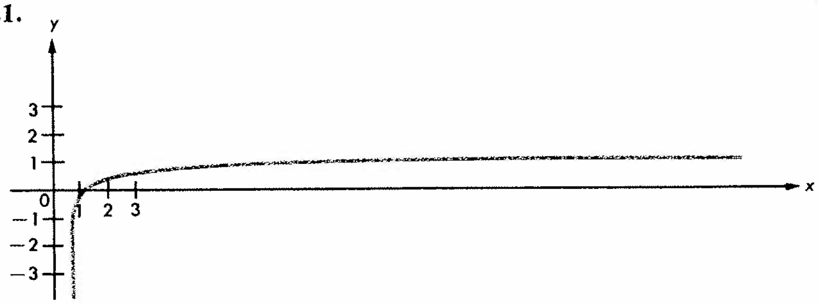
19.



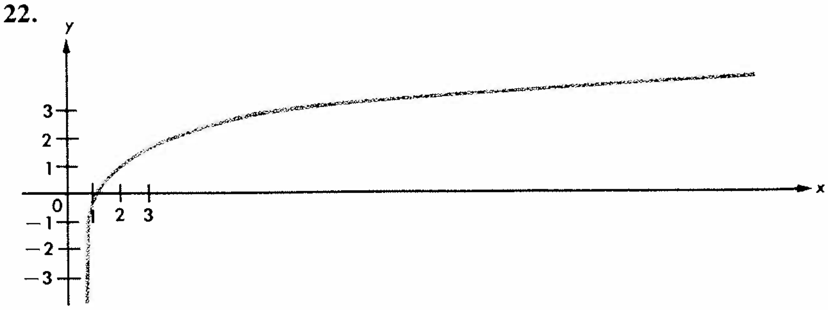
20.



21.



22.



23. $y = 2$ **24.** $y = 3$

25. $y = \frac{3}{2}$ **26.** $y = \frac{7}{2}$

7-5 COMMON LOGARITHMS

There are two bases for logarithms that are extensively used today. One is the base e ($e \doteq 2.71828$) widely used in higher mathematics. Logarithms to the base e are called *natural logarithms*. The other is the base 10, chosen because of our decimal system of notation. Logarithms to the base 10 are called *common logarithms*. We shall discuss only common logarithms in this section. Whenever we write $\log N$ without indicating the base, it is always understood that the base is 10; that is,

$$\log N \quad \text{means} \quad \log_{10} N.$$

By the very definition of logarithms,

$$\log 10^n = n$$

for every real number n. Thus, for example,

$$\log .01 = -2 \quad \text{because} \quad .01 = 10^{-2},$$
$$\log .1 = -1 \quad \text{because} \quad .1 = 10^{-1},$$